*Journal of Engineering Physics and Thermophysics, Vol. 81, No. 3, 2008*

# ANALYTICAL METHOD FOR ESTIMATING THE NONLINEAR PROPERTIES OF LIQUIDS BY A TORSION VISCOSIMETER

**I. V. Elyukhina and G. P. Vyatkin** UDC 532.137.3

*This paper considers the motion of a torsion viscosimeter that permits a conclusion about the rheological type of the liquid filling it. We have developed a theory of identification of nonlinear properties based on the use of the effective values of viscosity for each half-period and analytical solutions for linear liquids. The questions of the sensitivity that should be taken into account in planning an optimal experiment have been highlighted.*

**Introduction.** The method of torsional vibrations seems to be the most suitable for studying the viscous properties of liquid metals [1]. The conditions realized in the torsion viscosimeter — the change with time in the regime of medium deformation and the possibility of realizing ultimate small deformations and their rates — enable one to make visible even weak Newton effects in media usually considered to be Newtonian liquids (see also [2]). Moreover, here the conclusion about the rheological character is based on measurements of the vibration parameters which can be taken with a high accuracy (~10–5) impossible for other methods.

Despite the recent enhanced interest in the possibility of a non-Newtonian behavior of melts, similar investigations have been carried out only episodically [3, 4], and the area of limit low rates of shear from this standpoint have not been investigated at all. In the torsion-vibration method, results are traditionally interpreted from the point of view of the Newtonian model, which, in particular, is due to the lack of theory. The known solutions of the conjugate problem on the motion of a cylinder and a nonlinear specimen [5] have been obtained numerically, since, first of all, the change of sign of the deformation rate in the course of time and along the spatial coordinate makes it impossible to find an exact solution, which hampers practical use of results. Therefore, we shall present an analytical method for estimating the properties, which does not require the execution of numerical procedures, with the example of rheostable liquids. We shall present the results for the case of a long cylinder.

**Formulation of the Problem.** Let a hollow circular cylinder be suspended along its axis by an elastic cord and execute torsional vibrations with period $\tau_0$ and decay decrement $\delta_0$. When the crucible is filled with the liquid, as a consequence of its attraction by the walls moving with acceleration, the effective moment of inertia of the suspended system and the period of vibrations increase: $\tau > \tau_0$; there is also an increase in the rate of their damping because of the additional dissipation of mechanical energy caused by the viscous friction: $\delta > \delta_0$. The problems are as follows. The direct problem is to determine the properties of the liquid and the inverse problems is to predict the law of vibrations of the vessel and its conjugate: the motion of the viscosimeter is directly associated with the liquid motion initiated by it. Mathematical introduction into the method of measurements was made, in particular, in [1, 5, 6]. We shall describe the experimental conditions by the dimensionless complexes $A = MR^2/2K$, $\xi_0 = R/d$.

**Linear Liquids.** Note some of the features of estimating the properties of linear media, which is also useful in interpreting results for nonlinear media. A cylinder filled with linear, viscous or viscoelastic media upon completion of the transient process executes regular vibrations: $\alpha(T) = \alpha_0 \exp(-sT)$. The properties of such liquids can be described in terms of complex viscosity (e.g., [7]) and estimated from the quality function minimum condition

$$f = \sqrt{c_{Re} F_{Re}^2 + c_{Im} F_{Im}^2}\,, \qquad (1)$$

South-Ural State University, 76 V. I. Lenin Ave., Chelyabinsk, 454080, Russia; email: susu@74.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 81, No. 3, pp. 518–523, May–June, 2008. Original article submitted December 7, 2006.

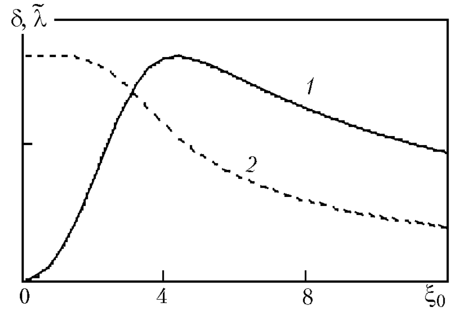



Fig. 1. Dependences $\delta = \delta(\xi_0)$ (1) and $(\lambda^2 - 1) = \tilde{\lambda}(\xi_0)$ (2).

$$F = 4A\sqrt{s}\,J_2(\sqrt{s}\,\xi_0\sqrt{\gamma}) + \varphi\xi_0\sqrt{\gamma}\,J_1(\sqrt{s}\,\xi_0\sqrt{\gamma})\,, \tag{2}$$

where $\varphi = \Delta\lambda^{-1}\left[1 + (\lambda^{-2} + \lambda^{-2}\Delta^2)^{-1}\right] + i\lambda^{-1}\left[1 - (\lambda^{-2} + \lambda^{-2}\Delta^2)^{-1}\right]$; $\delta_0 = 0$; $\gamma = 1$, $\gamma = 1 - s\mathrm{We}$ and $\gamma = [1 - 1/(s\mathrm{We})]^{-1}$ for Newtonian [1, 6], Maxwell, and Voigt liquids, respectively.

Relations taking into account the transient processes in Newtonian liquids are given in [6], where it has been shown that besides the fundamental harmonic there are aperiodic components. The irregular processes in viscoelastic media can be investigated in terms of analogous dependences [6], in which instead of $\xi_0$ for Maxwell and Voigt liquids the expressions $\xi_0\sqrt{(1 + S\mathrm{We})}$ and $\xi_0\sqrt{[1 + 1/(S\mathrm{We})]^{-1}}$ are given and the roots of $S$ have values other than for a viscous liquid, including complex ones, and $S = -s$ corresponds to regular vibrations.

For a Newtonian liquid [1] the dimensionless period $\lambda$ decreases with increasing $\xi_0$ mainly on the interval $\xi_0 \in (2; 12)$, and the behavior of $\delta$ depends on $\xi_0$: at $\xi_0 > \xi_{0\delta}$ the decrement decreases with increasing $\xi_0$, and at $\xi_0 < \xi_{0\delta}$ it increases (Fig. 1). For a long cylinder, $\xi_{0\delta} \sim 4.3$, and with decreasing height this peak of the function $\delta = \delta(\xi_0)$ shifts towards large values of $\xi_0$. Then in the region of this maximum the sensitivity of viscosity to the decrement and, consequently, also the error in $\nu$ due to the $\delta$ measurement errors are very high, as is the sensitivity to errors in a period at $\xi_0$ values close to zero and high values of $\xi_0$. As $A$ decreases, the curves in Fig. 1 become more sloping, i.e., the sensitivities, along with the errors that are due to one and the same measurement accuracy, increase. According to [1], on the left of $\xi_{0\delta}$ the approximation is highly viscous, at $\xi_0 > 0$ it is poorly viscous, and the interval between them corresponds to the intermediate viscosity. As the elastic properties strengthen, the number of extrema on the $\xi_0$ dependences of vibration parameters increases, which is determined by the length ratio between the viscous and the elastic waves.

The results of the calculations in terms of the sensitivity theory for viscous media can be found, for example, in [8]. For studying the viscoelastic properties, the method of forced vibrations [9] is preferable, and the method of damped vibrations is poorly effective for investigating, first of all, slightly elastic properties because of the narrow range of $\xi_0$ values. An adequate region for simultaneous measurement of viscous and elastic properties is the region where the values of the viscous and the elastic waves are close. Then the sensitivities $\psi_{y,x} = \left|(x/y)\,\partial y/\partial x\right|$ of the properties $y$ to errors in the parameters of the facility and vibrations $x$ are minimal and, for example, $\psi_{\mathrm{We},A} \sim 0.1, ..., 1$, and as the We number decreases by an order of magnitude, the value of $\psi_{\mathrm{We},A}$ increases by a factor of 5–10 and then by about an order of magnitude each time the We number further decreases by an order of magnitude. In the calculation by $F_{\mathrm{Im}}$, the sensitivity $\psi_{\mathrm{We},\lambda}$ is usually somewhat higher, as it is at We = 0, and with increasing We the number of local minima $f$ (1) increases. Therefore, in the presence of errors in the data $\Delta_x$ even for a viscous medium one can obtain We values close to possibly observable but far from correct ones. For instance, at $A = 0.15$ $\xi_0 = 12$ and $\Delta_{\xi_0} \sim 10^{-3}$, $\Delta_A \sim 10^{-2}$, $\Delta_\tau \sim 10^4$, $\Delta_\delta \sim 10^{-2}$ we obtain We $\sim 10^2$.

**Nonlinear Liquids.** *Specific features of the hydrodynamics.* The regularities in the flow of a liquid determining the response of vibration parameters to its rheological properties can be explained graphically by means of a model of a nonlinearly viscous medium, e.g., when the effective viscosity is equal to $bD^{m-1}$. At $m < 1$ (pseudoplastic liquids), the flow curves have a convexity upwards, to the stress axis, at $m > 1$ (dilatant liquids) they have a convexity downwards, and for a Newtonian liquid, at $m = 1$ these are straight lines. Under the viscosimeter conditions the rate of shear $D < 1$ and the effective viscosity for $m < 1$ has a value higher than for $m = 1$ prior to the point of intersection of these flow curves (at $D = 1$), and with increasing $D$ it becomes lower [2]. The penetration depth of the viscous wave is proportional to this viscosity and for a Newtonian liquid it is $\sim 10d$, and, e.g. for media with $m < 1$ and $D < 1$ it decreases with increasing $D$ and decreasing $m$. It should be noted that the boundary of the domain where the velocity is close to zero is closer to the wall at larger values of $m$. All other things being equal, $\tilde{D}$ is the larger the higher the value of $m$. As the elastic properties strengthen, the flow occurs in a larger volume, and the velocity profile approaches a rectilinear one.

When the viscosimeter is filled with a Bingham liquid [5], a solid core is always present near its axis, where the shear stresses do not exceed the yield stress. In the flow, there is also a thin solid layer (or several layers) that arises near the cylinder surface and moves to the core. In so doing, the core boundary moves from the center and then the zones merge. In the next quarter of the period reverse motion occurs, and so on. Here the region of the developed flow increases compared to the Newtonian liquid as a result of the entrainment of the liquid by the solid layers, which qualitatively agrees with the result for pseudoplastic media.

*Identification of the type of liquid.* The experimental identification of the rheological belonging of nonlinear media is based on the experimentally observed features associated with a break of isosynchronism of the vibration process. For such media, we shall determine the vibration parameters for each half-period:

$$\tau = 2t_\tau\,, \quad \delta = 2\ \ln\ |\alpha_1/\alpha_2|\ . \tag{3}$$

The regularities in the behavior of $\tau$ and $\delta$ depending on $\xi_0$ for Newtonian liquids (see Fig. 1) are also confirmed for the case of nonlinear viscous media if we consider, as $\xi_0$, the effective value $\xi_{0\text{eff}}$, which for such a liquid is $\xi_{0\text{nv}} = \xi_0/\sqrt{b\tilde{D}^{m-1}}$ and the value of $\xi_{0\text{nv}\delta} \sim 4.3$. With increasing vibration number $N$ the amplitude value of $\tilde{D}$ decreases. Therefore, for dilatant media the effective viscosity decreases and $\xi_{0\text{nv}}$ increases, and for pseudoplastic media — vice versa, and in the course of time $\lambda$ and $\delta$ change accordingly. Thus, for a dilatant medium, if $\xi_{0\text{nv}}$ at the beginning of vibrations corresponds to the low-viscosity approximation, then the vibration parameters decrease in the course of time, and if it corresponds to the highly viscous approximation, then $\delta = \delta(N)$ passes through the maximum. At $\xi_{0\text{nv}}$ close to $\xi_{0\text{nv}\delta}$ high values of $\delta$ are realized and vibrations decay faster than the dependence $\delta = \delta(\xi_{0\text{eff}})$ begins to decrease. For Newtonian liquids, at high $\xi_0$ with increasing $\xi_0$ the period decreases to a lesser extent than the decrement, and for dilatant media the change to the asymptotic regime characterized by slightly varying with time values is also faster for $\lambda$ than for $\delta$.

In the process of damping, the region of solid flow in viscoplastic liquids grows and then fills the entire viscosimeter at any instant of time. In this case, the effective moment of inertia of the systems reaches its largest value equal to the sum of the moments of inertia of the frozen liquid and the empty suspended system, and, along with it, the period directly proportional to the square root of this value also reaches its largest value: $\lambda_{\text{sf}} = (1 + A)^{1/2}$. Then the value of $\delta$ is minimal and coincides with $\delta_0$ because of the absence of mechanical energy dissipation due to the viscous friction. If from the beginning of vibrations an effective value of $\xi_{0\text{vp}} < \xi_{0\text{vp}\delta}$ is realized, then this function decreases. With decreasing $\xi_{0\text{eff}}$ in the interval corresponding to highly viscous media the value of $\delta$ decreases more significantly than increases, down to $\xi_{0\text{vp}\delta}$.

For elastic viscoplastic media, on the dependences of the parameters of vibrations depending on their number, minima can be observed. This is explained from the viewpoint of nonlinear models by the presence of one extremum on the dependence $\delta = \delta(\xi_0)$ for a Newtonian liquid and the increase in their number with strengthening elastic properties, whereas with increasing $N$, $1/\tilde{D}$, and effective viscosity $\xi_{0\text{eff}}$ decreases and some $\text{We}_{\text{eff}}$ increases. On the curves of $\delta = \delta(N)$ and $\lambda = \lambda(N)$ there is a jump at a nonanalytical behavior of the flow caused, e.g., by the existence of the medium at equal rates of shear or stresses in various microscopic states.

*Estimation of the liquid properties.* The specific features of vibration processes serve as the basis in choosing the rheological type, within which in the general case, according to the law of vibrations, one determines the properties of a liquid by the methods of parametric identification that make it possible to establish observability and identifiability of a system, find covariance matrices of errors of parameters being evaluated by such for quantities being measured, design an optimal experiment, etc. [10]. For estimation, one can use some special techniques based, in particular, for viscoplastic media on the observation of the number of vibrations before the solid regime sets in, and for nonlinear viscous media — on measuring the damping period and decrement at the beginning of the vibration process and asymptotic values at its end [5]. We turn our attention to the algorithm of the analytical method.

1. Let the input data be known from an experiment, whose conditions are as follows: a) the facility parameters: $A$, $\xi_0$; b) the vibration parameters: the frequency $\omega$ and the decrement $\Delta$ determined by the half-period $n = N/2$; the initial shift of the crucible: $\alpha_0 \sim 6^{\circ} = 0.1047$.

2. The values of $\alpha_{0n}$ are determined for each $n$ by $\alpha_0$ and $\delta = \delta(n)$: $\alpha_{0n+1} = \alpha_{0n}/\exp(\delta_n/2)$. In another variant, these values are known from the experiment.

3. From the viscosimetric equation (2)

$$F(\xi_{0\text{eff}}) = 4A\sqrt{s}\,J_2(\sqrt{s}\,\xi_{0\text{eff}}) + \varphi\xi_{0\text{eff}}J_1(\sqrt{s}\,\xi_{0\text{eff}}) = 0\,, \tag{4}$$

where for the Bessel functions the integral representations from [11] can be used, the effective values $\xi_{0\text{eff}}$ are estimated by $\Delta$ and $\omega$ for each half-period.

4. According to the dependence for the rate of shear for the Newtonian liquid

$$D = \left|\text{Re}\left\{-\xi_0\alpha_0 s\sqrt{s}\left[J_2(\sqrt{s}\,\xi_0)/J_1(\sqrt{s}\,\xi_0)\right]\exp(-sT)\right\}\right|\,, \tag{5}$$

for nonlinear liquids we obtain

$$\tilde{D} = \left|\text{Re}\left[-\xi_{0\text{eff}}\alpha_0 s\sqrt{s}\,\frac{J_2(\sqrt{s}\,\xi_{0\text{eff}})}{J_1(\sqrt{s}\,\xi_{0\text{eff}})}\,k\right]\right|\,,\quad k = \Delta\,\frac{1-\exp(-\Delta\pi)}{\pi(\Delta^2+1)} + 2\,\frac{\exp(-\Delta\pi/2)}{\pi(\Delta^2+1)}\,, \tag{6}$$

where $\alpha_0$, $s$ (including $\omega$ and $\Delta$), and $\xi_{0\text{eff}}$ are determined for each $n$.

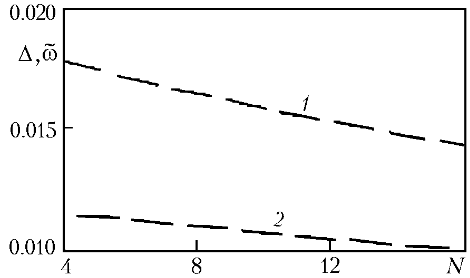
In view of the expression for $\xi_{0\text{eff}}$ and (6), e.g., for estimating $b$ and $m$, we solve the extreme problem:

$$f_{\text{eff}}(b, m) = \sqrt{\sum_n \left(\xi_0/\sqrt{b\tilde{D}_n^{m-1}} - \xi_{0\text{nv}n}\right)^2} \to \min\,, \tag{7}$$

where the form of the first term in the general case is determined by the rheological model.

5. In estimating the error in the nonlinear properties and the applicability limit of the method in the limiting cases of high- and low-viscous media, it is necessary to take into account the sensitivity of properties to errors in experimental data [5]. At a small number of experimental points function (7) has a ravine on the plane ($b$, $m$) with a slight change in its values along the axis, where local minima can be observed; the number of these minima can be decreased by changing $A$. Such a situation can result, in particular, from an insignificant change in $\omega$ and $\Delta$ in the damping process, and then, e.g., for dilatant media one should decrease $\xi_{0\text{nv}}$ at $N \to 1$. Note that the error in the estimated value of $\xi_{0\text{nv}}$ can reach tens of percent [1], and errors in the decrement can lead to errors in the $\alpha_{0n}$ values at large $n$ of a few percent if they are determined by $\alpha_0$ at $T = 0$ rather than from the experiment. The method is adequate for describing viscoplastic media when in the solid regime $\xi_{0\text{vp}} \sim 0$ and upon some modification can be used to investigate viscoelastic properties.

It has been established from the numerical calculations in terms of the sensitivity theory that in the presence of 20–50 measurement points and a change in the process of damping in the vibration parameters, usually the decrement, by 20% on average, one can achieve an error in the liquid properties of no more than 10% (the measurement accuracy of the period and decrement therewith is no worse than 0.1%), also taking into account the general recommendations on the sensitivity of $\xi_0$ to $\Delta$ and $\lambda$ for Newtonian liquids. The value of the rate of shear (6) reflects the effective value, which permits taking into account by default the features causing difficulties otherwise. Thus, the law $D = D(T)$ for nonlinear media differs from the harmonic law, also for each $n$, and on its spectrum at a pronounced nonlinearity the peak intensity ratio is $I_{3\omega}/I_{\omega} \sim 0.1$, and for Bingham in $\tilde{D}$ it is necessary to take into account the stage of vibrations in the presence of a solid zone near the wall. In the general case, the error in $\tilde{D}$ is equivalent to the error in $\alpha_{0n}$ and its influence can be elucidated within the sensitivity $\psi_{b,\alpha_0}$, $\psi_{m,\alpha_0}$, and so on.



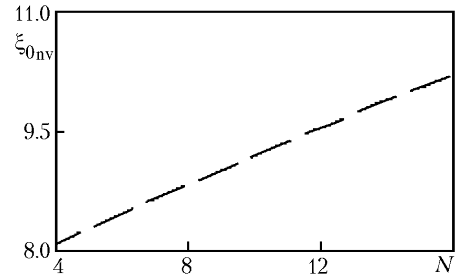




Fig. 2. Dependences of vibration parameters $(1 - \omega) = \tilde{\omega}(N)$ (1) and $\Delta = \Delta(N)$ (2) for dilatant liquids.

Fig. 3. Dependence $\xi_{0nv} = \xi_{0nv}(N)$.

In the case of a finite-height crucible, the dependence for the rate of shear in (5) also takes into account the other components of the tensor of deformation rates, in particular, under traditional assumptions the expression for the azimuthal velocity from [1, 6] is used. The dependences can be used to solve the inverse problem and in investigating the identifiability of properties and choosing optimal conditions. Then for a more accurate description of the crucible motion it is necessary to take into account the transient processes with the help of relations for a linear medium constructed in terms of $\xi_{0eff}$ with a large number of points for $s$, $\tilde{D}$, etc. by determining these characteristics locally rather than by the half-period. In using the method for the regime of forced vibrations, in relation (6) the quantities $k = 2/\pi$, $s = i\omega$, $\alpha_0$ is the vibration amplitude of the crucible, and the experimental points are selected, e.g., from the dependences of the amplitude and phase on the vibration frequency.

**Examples.** *Nonlinear viscous media.* We are given: $A = 0.1$, $\xi_0 = 8$, $\alpha_0 = 0.1047$; the change in the vibration parameters in the damping process (Fig. 2) without account for the transient processes (obtained by the numerical simulation [5] for a medium with $b = 2.4$, $m = 1.8$); the error in the vibration law: $\Delta_\Delta$, $\Delta_\omega \sim 0.1\%$, it is not taken into account in the other data. It is required to determine the liquid properties.

We determine by the values of $\Delta$ (Fig. 2) the initial angular displacement amplitude in each half-period $\alpha_{0n}$, and by $\Delta$ and $\omega$ (Fig. 2) the effective values of $\xi_{0nv}$ (Fig. 3). Judging by the character of the change in the vibration parameters and $\xi_{0nv}$, the liquid belongs to the nonlinear viscous type, and, more precisely, to dilatant media. Estimating $b$ and $m$ by relation (7), we obtain $b = 2.507$ and $m = 1.843$. Such a "good" situation could have resulted from a combination of errors and, therefore, it is necessary to estimate their sensitivities to the facility and vibration parameters. Thus, the values of $\psi_{m,\xi_{0nv}}$ and $\Psi_{m,\alpha_0}$ determined by (7) are small, and those of $\psi_{b,\xi_{0nv}}$ and $\psi_{b,\alpha_0}$ are of the order of one. In estimating $\xi_{0nv}$, the sensitivity $\psi_{\xi_{0nv}\omega}$ determined by (4) is ~50 on average, which yields an error in $\xi_{0nv}$ of ~5%. The largest deviation from the ideal value is, as in the traditional theory, due to the high sensitivity to the frequency determined in terms of $F_{Im}$ (4) (in the region of the peak of the function $\Delta = \Delta(\xi_0)$ the values of $\psi_{\xi_{0nv,\Delta}}$ are also high). Note that the quality function $F_{Re}$ is usually more sloping, i.e., the influence of errors in the viscosimeter parameters for it is stronger, and the more pronounced minimum of the function $F_{Im}$ undergoes a greater shift as a consequence of errors in the frequency. In the given example (where the values of $\psi_{\xi_0,\omega}$ are not high), it is expedient to perform calculations with account for both parts of the viscosimetric function. Taking into account the obtained values of $\psi_{y,x}$ and the accuracy of the parameters entering into (7), we obtain that the maximum admissible errors in the values of $b \sim 10\%$ and $m \sim 3.5\%$, and the $b$ and $m$ estimates fall within these intervals.

At $m$ not close to 1 the quality function can be formed by the $\tilde{D}_n$ values: by (6) and $[(\xi_0^2/\xi_{0nvn})/b]^{1-m}$. According to the analysis of the sensitivity, in this case smaller interval estimates can be obtained, as, e.g., in the given calculation. Performing minimization, we obtain $b = 2.305$ and $m = 1.836$. In both cases, we have used for the estimation all experimental points for the $N$ values given in Fig. 2.

*Viscoplastic media.* We are given: $A = 0.2$, $\xi_0 = 12$, $\alpha_0 = 0.1047$; the relations $\Delta = \Delta(N)$ and $\omega = \omega(N)$ (Fig. 4: for a medium with $b = 1$, Bm = 0.4 [5], where the effective viscosity in the viscoplastic flow is $b(1 + \text{Bm}/D)$); $\Delta_\Delta$, $\Delta_\omega \sim 0.5\%$. It is required to determine $b$ and the Bingham number.

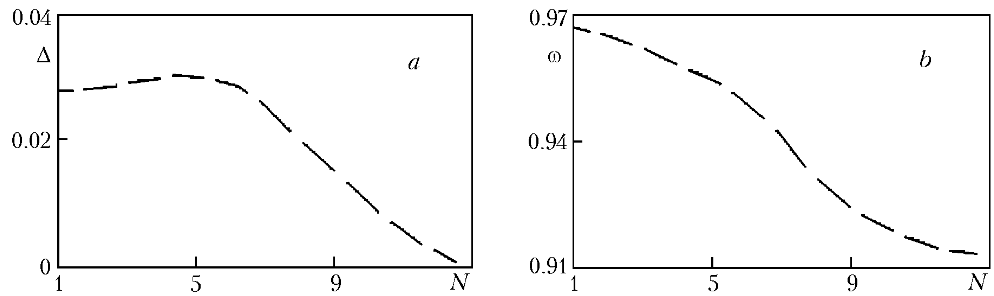



Fig. 4. Logarithmic decrement (a) and vibration frequency (b) versus their number for Bingham liquids.

A decrease in the frequency in the damping process points to a (visco- and pseudo-) plastic type of the liquid. An abrupt, rather than asymptotic, change to the isosynchronous operation, when, in addition, $\omega \sim 1/\sqrt{1+A}$, and $\Delta \sim 0$, shows that the liquid has a yield point. In view of the expression for $\xi_{0vp}$ function (7) changes adequately. The values of the properties corresponding to the minimum $f_{eff}$ deviate from the ideal ones by less than 20% ($b$ = 0.839, Bm = 0.332) and correspond to the confidence intervals obtained from the point of view of sensitivity. In describing the behavior by the nonlinear viscous model, we obtain $b \sim 1$ and $m \sim 0.25$, and the goal function, even at $N \in [1; 12]$, i.e., without account for the isosynchronous operation, takes on values in the optimum higher by an order of magnitude (and has local minima), which confirms the correctness of choosing the equation of state for the viscoplastic liquid.

Part of this work was supported by the Russian Foundation for Basic Research — Ural (No. 07-02-96016).

## NOTATION

$A$, ratio between the moments of inertia of the liquid in the viscosimeter and the empty suspended system; Bm = $\sigma_0/(\nu\rho q_0)$, Bingham number; $b = q_0^{m-1}k/\nu\rho$, dimensionless consistency index; $c_{Re}$, $c_{Im}$, weight factors; $d = \sqrt{\nu/q_0}$, natural length scale, thickness of the boundary layer, m; $D$, dimensionless rate of shear; $\tilde{D}$, half-period-averaged value of $D$ at $\xi = \xi_0$; $F$, viscosimetric function; $f$, quality function; $G$, shear modulus, Pa; $I_{l\omega}$, intensity of harmonic $l\omega$ ($l$ = 1, 2, ...); $i = \sqrt{-1}$, imaginary unit; $J_{1,2}$, first- and second-order Bessel functions of the first kind; $k$, constant of the power rheological law, kg·sec$^{m-2}$/m; $K$, moment of inertia of the empty suspended system, kg·m$^2$; $M$, mass of the liquid, kg; $m$, exponent of the power rheological law; $N$, vibration number; $n$, half-period number; $q_0 = 2\pi/\tau_0$, vibration frequency of the empty cylinder, rad/sec; $R$, internal radius of the cylinder, m; $S$, pre-exponential factor in the vibration law; $s = (\Delta + i)/\lambda$, factor corresponding to the regular regime; $T = q_0 t$, dimensionless time; $t$, time, sec; $t_\tau$, difference between two adjacent instants of time when $\alpha \sim 0$, sec; We = $\theta q_0$, Weissenberg number; $\alpha$, angular displacement of the crucible from the equilibrium position, rad; $\alpha_1$, $\alpha_2$, adjacent extreme values of $\alpha$ ($|\alpha_1| > |\alpha_2|$); $\gamma$, $\varphi$, coefficients of the viscosimetric equation taking into account the influence of liquid properties and vibration parameters; $\delta$, logarithmic decrement of cylinder vibrations at $M \neq 0$; $\delta_0$, logarithmic decrement of empty cylinder vibrations; $\Delta = \delta/(2\pi)$, logarithmic decrement; $\Delta_x$, error in the parameter $x$ measurement; $\lambda = \tau/\tau_0$, dimensionless vibration period; $\nu$, kinematic viscosity of the medium, m$^2$/sec; $\theta = \rho\nu/G$, relaxation (or delay) time, sec; $\rho$, medium density, kg/m$^3$; $\sigma_0$, yield point, kg/(m·sec$^2$); $\tau$, vibration period of the cylinder at $M \neq 0$, sec; $\tau_0$, natural period, sec; $\omega = 1/\lambda$, dimensionless vibration frequency; $\xi_0$, ratio of the radius to the thickness of the boundary layer; $\psi_{y,x}$, sensitivity of the parameter $y$ to $x$. Subscripts: 0, natural vibrations as well as initial-boundary conditions: $\xi_0$, on the cylinder wall; $\alpha_0$, at the initial instant of time; Re and Im, real and imaginary parts; $\delta$, maximum of $\delta = \delta(\xi_0)$; vp, viscoplastic; nv, nonlinear viscous; sf, solid flow; eff, effective.

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